DETERMINANTS

PART 1

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TOPICS

- DEFINITION AND EVALUATION OF DETERMINANTS(ROW WISE &COLUMN WISE)
- PROPERTIES OF DETERMINANTS
- AREA OF TRIANGLE
- MINORS AND CO-FACTORS OF DETERMINANTS
- ADJOINT AND INVERSE OF A MATRIX
- APPLICATION OF MATRICES AND DETERMINANTS

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INTRODUCTION

In the previous chapter, we have studied about matrices and algebra of matrices. We have also learnt that a system of algebraic equations can be expressed in the form of matrices. This means, a system of linear equations like

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

can be represented as $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

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DEFINITION

To every square matrix $A = [a_{ij}]$ of order n, we can associate a number (real or complex) called determinant of the square matrix A, where $a_{ij} = (i, j)^{th}$ element of A.

A DETERMINANT IS A FUNCTION FROM A SET OF SQUARE MATRICES TO A SET OF REAL OR COMPLEX NUMBERS

If M is the set of square matrices, K is the set o numbers (real or complex) and $f : M \to K$ is defined by $f(A) = k$, where $A \in M$ an $k \in K$, then $f(A)$ is called the determinant of A. It is also denoted by $|A|$ or det A or Δ

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then determinant of A is written as $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det(A)$

Remarks

(i) For matrix A, $|A|$ is read as determinant of A and not modulus of A.

(ii) Only square matrices have determinants.

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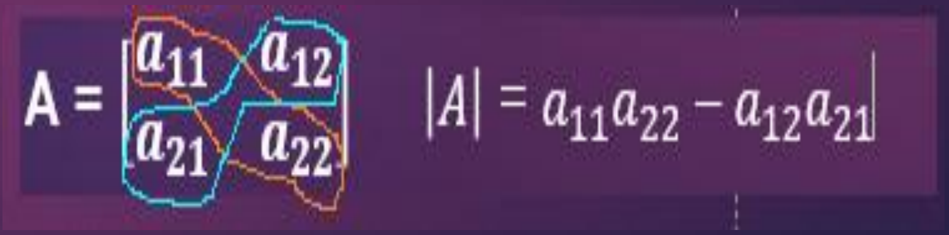
DETERMINANT OF MATRIX OF ORDER 1 & 2

- **ORDER 1**
- A = [a], det A = a
- Let A = [4], A Is a Square Matrix of order 1 x 1 (order1).

  Then det A = 4   OR We can write $|A|$ = 4   OR Δ = 4
- Let A = [-4], A Is a Square Matrix of order 1 x 1 (order1).

  Here also, det A = 4   OR We can write $|A|$ = 4   OR Δ = 4
- **ORDER 2**
- **A =** $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $|A|$ = ad – bc
- **A =** $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $|A| = a_{11}a_{22} - a_{12}a_{21}$



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Eg: Evaluate a determinant order 2

- Eg: If A = $\begin{bmatrix} 4 & 5 \\ 3 & -1 \end{bmatrix}$, find $|A|$. $|A|$ = 4 x (-1) – 5 x 3 = -19
- If P = $\begin{bmatrix} c & d \\ -5 & 2 \end{bmatrix}$, find $|P|$. $|P|$ = 2c – (-5d) = 2c + 5d
- Δ = $\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix}$ Evaluate

  = 1x2 – 4x2 = 2 - 8 = - 6

ISM

DETERMINANT OF A MATRIX OF ORDER 3 X 3

- $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

- EXPANSION ALONG ROW 1 (R_1)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A| = (-1)^{1+1} a_{11} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- EXPANSION ALONG COLUMN 2 (C_2)

$$|A| = (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{2+2} a_{22} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{3+2} a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

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DETERMINANT OF A MATRIX OF ORDER 3 X 3

EXPANSION ALONG ROW 1 (R_1)

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

ISM

working

- Eg : If A = $\begin{pmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{pmatrix}$ Expand along R_1

$=(-1)^{1+1}$x3(1+6)+$(-1)^{1+2}$x -4(1+4)+$(-1)^{1+3}$x5(3-2)

= + 21 – (-20) + 5 = 46

- Expand along C_2

= $(-1)^{1+2}$x(-4)(1+4)+ $(-1)^{2+2}$x 1(3-10)+$(-1)^{3+2}$ x 3(-6-5)

= - (-20) + (-7) – (-33) = 46

REMARK (1)

- While Expanding , instead of multiplying by $(-1)^{i+j}$, we can multiply by (+1) or (-1) according as (I + j) is even or odd.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

- Evaluate $\Delta = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{vmatrix}$

Expand along C_1

Δ = +1 (-9+12) – 2 (-9 + 8) +5 (-3 + 2)

= 3 + 2 – 5 = 0

Remark (2)

- For easier calculations, we shall expand the determinant along that row or column which contains maximum number of zeros.

Eg: Evaluate $\Delta = \begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix}$

- Note that in the third column, two entries are zero. So expanding along third column (C_3)

$$\Delta = +4\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}$$

$= 4 (-1 – 12) + 0 + 0$

$= - 52$

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Remark (3)

- $|k\,A| = k^n\,|A|$ [n is the order of the matrix]

Eg: A = $\begin{pmatrix} 1 & 2 \\ 4 & 2 \end{pmatrix}$

$|2A| = 2^2|A| = 4 \times (2 - 8) = -24$

$|3A| = 3^2|A| = 9 \times (2 - 8) = -54$

Eg: A = $\begin{pmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{pmatrix}$ $|A| = 46$

$|2A| = 2^3|A| = 8 \times 46 = 368$

$|3A| = 3^3|A| = 27 \times 46 = 1242$

Equality of Determinants

- Eg : $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$ then find the value of x .

  $x^2 - 36 = 36 - 36$

  $x^2 = 36$

  $x = \pm 6$

- HOME WORK

EX : 4 .1 - Q 1 , 2 , 5 , 7

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DETERMINANTS-PART 2

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PROPERTIES OF DETERMINANTS

1. $|A^T| = |A|$

Eg: A = $\begin{pmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{pmatrix}$ $|A|$**= + 21 - (-20) + 5 = 46**	A^T **=** $\begin{pmatrix} 3 & 1 & 2 \\ -4 & 1 & 3 \\ 5 & -2 & 1 \end{pmatrix}$ $|A^T|$ **= 46**

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2

- If we interchange any two rows (or columns), then sign of the determinant changes
- (if we change twice , then the value remains same)

Eg: $A = \begin{pmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{pmatrix}$	$B = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & -2 \\ 3 & -4 & 5 \end{pmatrix}$
$\lvert A \rvert = + 21 - (-20) + 5$	$\lvert B \rvert = 2(5-8) -3(5+6)+1(-4-3)$
$= 46$	$= -46$

3.

- If any two rows or any two columns in a determinant are identical (or proportional), then the value of the determinant is zero.

$A = \begin{vmatrix} 1 & 4 & 0 \\ 2 & 0 & 5 \\ 1 & 4 & 0 \end{vmatrix}$	$A = \begin{vmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 1 & 2 & 8 \end{vmatrix}$
$= 1(0-20)-4(0-5)\ +0$	$= 1(32-10)\ -2\ (16-5)\ +0$
$= 0$	$= 0$

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4.

- If we multiply each element of a row (or a column) of a determinant by constant k, then value of the determinant is multiplied by k.

(Multiplying a determinant by k means multiplying the elements of only one

row (or one column) by k.)

$$Eg: \Delta = \begin{vmatrix} 1 & 4 & 0 \\ 2 & 0 & 5 \\ 2 & 4 & 0 \end{vmatrix} = 0 - 5(4-8) + 0 = 20$$

$$C_1 \rightarrow 3\,C_1$$

$$\begin{vmatrix} 3 & 4 & 0 \\ 6 & 0 & 5 \\ 6 & 4 & 0 \end{vmatrix} = 0 - 5(12 - 24) = 60$$

$$= 3 \times 20 = 3\,\Delta$$

$$\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

ISM

5.

If elements of a row (or a column) in a determinant can be expressed as the sum of two or more elements, then the given determinant can be expressed as the sum of two or more determinants.

$$\begin{vmatrix} a_1 + a_2 & b & c \\ p_1 + p_2 & q & r \\ u_1 + u_2 & v & w \end{vmatrix} = \begin{vmatrix} a_1 & b & c \\ p_1 & q & r \\ u_1 & v & w \end{vmatrix} + \begin{vmatrix} a_2 & b & c \\ p_2 & q & r \\ u_2 & v & w \end{vmatrix}$$

$$\begin{vmatrix} 50 & 2 & 7 \\ 40 & 9 & 0 \\ 80 & 7 & 2 \end{vmatrix} = \begin{vmatrix} 20 & 2 & 7 \\ 15 & 9 & 0 \\ 30 & 7 & 2 \end{vmatrix} + \begin{vmatrix} 30 & 2 & 7 \\ 25 & 9 & 0 \\ 50 & 7 & 2 \end{vmatrix}$$

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6.

- If to each element of a row (or a column) of a determinant the equimultiples of corresponding elements of other rows (columns) are added, then value of determinant remains same. $R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} + ka_{31} & a_{12} + ka_{32} & a_{13} + ka_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

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REMARKS:

- 1.If all the elements of a row (or column) are zeros, then the value of the determinant is zero

$$Eg: \quad \begin{vmatrix} 1 & 0 & 3 \\ 5 & 0 & 5 \\ 8 & 0 & 4 \end{vmatrix} = 0$$

- 2.If all the elements of a determinant above or below the main diagonal consists of zeros, then the value of the determinant is equal to the product of diagonal elements. Eg: $\begin{vmatrix} 5 & 0 & 0 \\ 5 & 9 & 0 \\ 8 & 7 & 2 \end{vmatrix} = 90$

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NOTES:

- A is a singular matrix, $|A| = 0$
- $|AB| = |A||B|$
- $|k\,A| = k^n\,|A|$ (n is the order of the matrix)
- $|A^n| = |A|^n$
- If A is a non-singular matrix, then $|A^{-1}| = 1 / |A| = |A|^{-1}$
- In general, $|B + C| \neq |B| + |C|$

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PROBLEMS

- 1)

Q1. Evaluate $\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$

R1 and R3 are proportional

Solution Note that $\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 6(17) & 6(3) & 6(6) \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = 6\begin{vmatrix} 17 & 3 & 6 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = 0$

Q2. Show that $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = 0$

Solution We have $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = \begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} + \begin{vmatrix} a & b & c \\ 2x & 2y & 2z \\ x & y & z \end{vmatrix}$

(by Property 5)

$= 0 + 0 = 0$ (Using Property 3 and Property 4)

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Q3.

Q3. Prove that $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3$.

Solution Applying operations $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$ to the given determinant Δ, we have

$$\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 3a & 7a+3b \end{vmatrix}$$

Now applying $R_3 \to R_3 - 3R_2$, we get

$$\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 0 & a \end{vmatrix}$$

Expanding along C_1, we obtain

$$\Delta = a\begin{vmatrix} a & 2a+b \\ 0 & a \end{vmatrix} + 0 + 0$$

$$= a\,(a^2 - 0) = a\,(a^2) = a^3$$

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4.

Q4. Without expanding, prove that

$$\Delta = \begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Solution Applying $R_1 \to R_1 + R_2$ to Δ, we get

$$\Delta = \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$$

Since the elements of R_1 and R_3 are proportional, $\Delta = 0$.

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Q5.

Q5. Evaluate

$$\Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

Solution Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} 1 & a & bc \\ 0 & b-a & c(a-b) \\ 0 & c-a & b(a-c) \end{vmatrix}$$

Taking factors $(b-a)$ and $(c-a)$ common from R_2 and R_3, respectively, we get

$$\Delta = (b-a)(c-a)\begin{vmatrix} 1 & a & bc \\ 0 & 1 & -c \\ 0 & 1 & -b \end{vmatrix}$$

$$= (b-a)(c-a)[(-b+c)] \text{ (Expanding along first column)}$$

$$= (a-b)(b-c)(c-a)$$

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HOME WORK

Using the property of determinants and without expanding in Exercises 1 to 7, prove that:

1. $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$

2. $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

3. $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0$

4. $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$

5. $\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$

EX. 4.2

Q 1,2,3,4,5

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DETERMINANTS (PART3)

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Q.6

Question 7:

By using properties of determinants, show that:

$$\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

Solution 7:

$$\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$$

$$= abc\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$ [Taking out factors a,b,c from R_1, R_2 and R_3]

$$= a^2b^2c^2\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$ [Taking out factors a,b,c from C_1, C_2 and C_3]

Applying $R_2 \to R_2 + R_1$ and $R_3 \to R_3 + R_1$, we have:

$$\Delta = a^2b^2c^2\begin{vmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix}$$

$$= a^2b^2c^2(-1)\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix}$$

$$= -a^2b^2c^2(0-4) = 4a^2b^2c^2$$

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Q.7

(ii) Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, we have:

$$\Delta = \begin{vmatrix} 0 & 0 & 1 \\ a-c & b-c & c \\ a^3-c^3 & b^3-c^3 & c^3 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-c & b-c & c \\ (a-c)(a^2+ac+c^2) & (b-c)(b^2+bc+^2) & c^3 \end{vmatrix}$$

$$= (c-a)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & c \\ -(a^2+ac+c^2) & (b^2+bc+c^2) & c^3 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2$, we have:

$$\Delta = (c-a)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ (b^2-a^2)+(bc-ac) & (b^2+bc+c^2) & c^3 \end{vmatrix}$$

$$= (b-c)(c-a)(a-b)\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ -(a+b+c) & (b^2+bc+c^2) & c^3 \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)(a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ -1 & (b^2+bc+c^2) & c^3 \end{vmatrix}$$

Expanding along C_1, we have:

T=(a-b)(b-c)(c-a)(a+b+c)

$$\Delta = (a-b)(b-c)(c-a)(a+b+c)(-1)\begin{vmatrix} 0 & 1 \\ 1 & c \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)(a+b+c)$$

Hence, the given result is proved.

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Q8

$$L.H.S = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} \qquad \textbf{PT} \quad \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Now by applying, $R_1 \to a R_1$, $R_2 \to b R_2$, $R_3 \to c R_3$

We get,

$$= \left(\frac{1}{abc}\right) \begin{vmatrix} a & a^2 & abc \\ b & b^2 & cab \\ c & c^2 & abc \end{vmatrix}$$

$$= \left(\frac{abc}{abc}\right) \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Hence, the proof.

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Q.9

Question 11:

By using properties of determinants, show that:

(i) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

(ii) $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$

Solution 11:

(i) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

Applying $R_1 \to R_1 + R_2 + R_3$, we have:

$$\Delta = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we have:

$$\Delta = (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix}$$

Expanding along C_3, we have:

$$\Delta = (a+b+c)^3(-1)(-1) = (a+b+c)^3$$

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Q.10

(ii) $\Delta = \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we have:

$$\Delta = \begin{vmatrix} 2(x+y+z) & x & y \\ 2(x+y+z) & y+z+2x & y \\ 2(x+y+z) & x & z+x+2y \end{vmatrix}$$

$$= 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have:

$$\Delta = 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 0 & x+y+z & 0 \\ 0 & 0 & x+y+z \end{vmatrix}$$

$$= 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along R_3, we have:

$\Delta = 2(x+y+z)^3(1)(1-0) = 2(x+y+z)^3$

Hence, the given result is proved.

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Q11

Q12) $\begin{bmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{bmatrix} = (1 - x^3)^2$, $x \neq 0$

Solution: C1 → C1 - x.C3

$$\begin{bmatrix} 1 - x^3 & x & x^2 \\ 0 & 1 & x \\ 0 & x^2 & 1 \end{bmatrix}$$

Expand along C1

= $(1- x^3)$ $(1- x^3)$

= $(1- x^3)^2$

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Q12.

We need to prove the following identity:

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$$

Let us consider the L.H.S of the above equation.

$$\Delta = \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$$

Applying $R_1 \to R_1(a)$, $R_2 \to R_2(b)$ and $R_3 \to R_3(c)$, we get

$$\Delta = \frac{1}{abc}\begin{vmatrix} a(a^2+1) & a^2b & a^2c \\ ab^2 & b(b^2+1) & b^2c \\ c^2a & c^2b & c(c^2+1) \end{vmatrix}$$

Taking a,b, and c common from C_1,C_2 and C_3, respectively, we get,

$$\Delta = \frac{abc}{abc}\begin{vmatrix} (a^2+1) & a^2 & a^2 \\ b^2 & (b^2+1) & b^2 \\ c^2 & c^2 & (c^2+1) \end{vmatrix}$$

Applying $R_1 \to R_1+R_2+R_3$, we get,

$$\Delta = \frac{abc}{abc}\begin{vmatrix} (a^2+b^2+c^2+1) & (a^2+b^2+c^2+1) & (a^2+b^2+c^2+1) \\ b^2 & (b^2+1) & b^2 \\ c^2 & c^2 & (c^2+1) \end{vmatrix}$$

Taking the term, $(a^2+b^2+c^2+1)$ common from the above equation, we have,

$$\Delta = (a^2+b^2+c^2+1)\begin{vmatrix} 1 & 1 & 1 \\ b^2 & (b^2+1) & b^2 \\ c^2 & c^2 & (c^2+1) \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we get,

$$\Delta = (a^2+b^2+c^2+1)\begin{vmatrix} 1 & 0 & 0 \\ b^2 & 1 & 0 \\ c^2 & 0 & 1 \end{vmatrix}$$

$$\Rightarrow \Delta = (a^2+b^2+c^2+1)$$

Q 13.

$$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} = 1$$

Solution:

Let $\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix}$

We know that the value of a determinant remains same if we apply the operation $R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$.

Applying $C_2 \to C_2 - pC_1$, we get

$$\Delta = \begin{vmatrix} 1 & 1+p-p(1) & 1+p+q \\ 2 & 3+2p-p(2) & 4+3p+2q \\ 3 & 6+3p-p(3) & 10+6p+3q \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 1 & 1+p+q \\ 2 & 3 & 4+3p+2q \\ 3 & 6 & 10+6p+3q \end{vmatrix}$$

Applying $C_3 \to C_3 - qC_1$, we get

$$\Delta = \begin{vmatrix} 1 & 1 & 1+p+q-q(1) \\ 2 & 3 & 4+3p+2q-q(2) \\ 3 & 6 & 10+6p+3q-q(3) \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 1 & 1+p \\ 2 & 3 & 4+3p \\ 3 & 6 & 10+6p \end{vmatrix}$$

Applying $C_3 \to C_3 - pC_2$, we get

$$\Delta = \begin{vmatrix} 1 & 1 & 1+p-p(1) \\ 2 & 3 & 4+3p-p(3) \\ 3 & 6 & 10+6p-p(6) \end{vmatrix}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1+p+q-q(1) \\ 2 & 3 & 4+3p+2q-q(2) \\ 3 & 6 & 10+6p+3q-q(3) \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 1 & 1+p \\ 2 & 3 & 4+3p \\ 3 & 6 & 10+6p \end{vmatrix}$$

Applying $C_3 \to C_3 - pC_2$, we get

$$\Delta = \begin{vmatrix} 1 & 1 & 1+p-p(1) \\ 2 & 3 & 4+3p-p(3) \\ 3 & 6 & 10+6p-p(6) \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, we get

$$\Delta = \begin{vmatrix} 1 & 1-1 & 1 \\ 2 & 3-2 & 4 \\ 3 & 6-3 & 10 \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 4 \\ 3 & 3 & 10 \end{vmatrix}$$

Applying $C_3 \to C_3 - C_1$, we get

$$\Delta = \begin{vmatrix} 1 & 0 & 1-1 \\ 2 & 1 & 4-2 \\ 3 & 3 & 10-3 \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 3 & 3 & 7 \end{vmatrix}$$

Expanding the determinant along R_1, we have

$\Delta = 1[(1)(7) - (3)(2)] - 0 + 0$

$\therefore \Delta = 7 - 6 = 1$

14.

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = ((1+a^2+b^2)^3$$

C1→ $C1 - bC3\,;C2 \rightarrow C2 + aC3$

$$\begin{vmatrix} 1+a^2+b^2 & 2ab & -2b \\ 0 & 1-a^2+b^2 & 2a \\ b+ba^2+b^3 & -2a & 1-a^2-b^2 \end{vmatrix}$$

$$1+a^2+b^2 \begin{vmatrix} 1 & 2ab & -2b \\ 0 & 1-a^2+b^2 & 2a \\ b & -2a & 1-a^2-b^2 \end{vmatrix}$$

R3 $\rightarrow R3 - bR1 + aR2$

$$1+a^2+b^2 \begin{vmatrix} 1 & 2ab & -2b \\ 0 & 1-a^2+b^2 & 2a \\ 0 & -a(1+a^2+b^2) & 1+a^2+b^2 \end{vmatrix}$$

=

$$((1+a^2+b^2)^2 \begin{vmatrix} 1 & 2ab & -2b \\ 0 & 1-a^2+b^2 & 2a \\ 0 & -a & 1+a^2+b^2 \end{vmatrix}$$

EXPAND ALONG C1

$$((1+a^2+b^2)^3$$

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HOME WORK

EX.4.2

Q 7,9,10, 11(ii)

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